MAULANA ABUL KALAM AZAD UNIVERSITY OF TECHNOLOGY, WEST BENGAL

M-101

MATHEMATICS-I

Time Allotted: 3 Hours Full Marks: 70

The questions are of equal value.
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

GROUP A
(Multiple Choice Type Questions)

1. **Answer any *ten* questions.** $10 \times 1 = 10$

(i) If 2, 3, 5 are the three eigenvalues of a 3rd order matrix A, then the value of det (A) is

(A) 30 (B) −30
(C) 0 (D) none of these

(ii) The matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is

(A) symmetric (B) skew-symmetric
(C) singular (D) orthogonal

(iii) 0 is an Eigenvalue of matrix if the matrix is

(A) non-singular (B) orthogonal
(C) skew-symmetric (D) singular

(iv) The function $f(x) = |x-2|$ satisfies Rolle's Theorem in the interval

(A) [3, 4] (B) [0, 4]
(C) [−3, 3] (D) [−1, 4]

(v) The value of $\int_0^{\frac{\pi}{2}} \sin^8 x\,dx = ?$

(A) $\frac{35}{128}$ (B) $\frac{35}{256}$
(C) $\frac{35\pi}{128}$ (D) $\frac{35\pi}{256}$

(vi) $\frac{\partial(x^y)}{\partial y} = ?$

(A) x^y (B) $x^y \log y$
(C) $x^y \log x$ (D) does not exist

(vii) If $x = r\cos\theta$ and $y = r\sin\theta$ then $\frac{\partial(r,\theta)}{\partial(x,y)} = ?$

(A) r (B) 1
(C) $\frac{1}{r}$ (D) none of these

(viii) The necessary condition that (a, b) is a ________ point of $f(x, y)$ if $f_x(a,b) = 0 = f_y(a,b)$

(A) maximum (B) stationary
(C) saddle point (D) minimum

(ix) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if

(A) $p < 1$ (B) $p > 0$
(C) $p > 1$ (D) $p < 0$

(x) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is

(A) absolutely convergent (B) conditionally convergent
(C) oscillatory (D) none of these

(xi) The value of λ for which the vector function $\vec{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+\lambda z)\hat{k}$ is solenoidal is

(A) -2 (B) 1
(C) 3 (D) 2

(xii) In the mean value theorem $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$, if $f(x) = \frac{1}{1+x}$ and $h = 3$, then the value of θ is

(A) 1 (B) $\frac{1}{3}$

(C) $\frac{1}{\sqrt{2}}$ (D) none of these

GROUP B
(Short Answer Type Questions)

Answer any *three* questions. $3 \times 5 = 15$

2. If $y = e^{m \sin^{-1} x}$, then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. 5
Find y_n for $x = 0$.

3. Using M.V.T. prove that $1 + \frac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1 + \frac{x}{2}$.

4. Expanding the determinant by Laplace's method in terms of minors of 2nd order formed from the first two rows.

5. Using M.V.T. prove that $x > \tan^{-1} x > \frac{x}{1+x^2}$, $0 < x < \frac{\pi}{2}$.

6. State D' Alembert's ratio test for convergence of an infinite series. Examine the convergence and divergence of the series.

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots\infty$$

GROUP C
(Long Answer Type Questions)

Answer any *three* questions. $3 \times 15 = 45$

7. (a) Expanding the determinant by Laplace's method in terms of minors of 2nd order formed from the first two, prove that

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + cd)^2.$$

(b) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) If $y = (x^2 - 1)^n$, then prove that $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$.

8. (a) If $u = \phi(x, y)$, and $x = r\cos\theta$, $y = r\sin\theta$ then if the variables are changed from x, y to r, θ then show that 6

(i) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

(b) Prove that the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots(-1)^{n+1}\frac{x^n}{n} + \ldots\ldots\infty$ is absolutely convergent when $|x| < 1$ and conditionally convergent when $x = 1$. 5

(c) Prove that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$. 4

9. (a) If $u = \tan^{-1}\frac{x^3 + y^3}{x - y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$ 5

(b) $\begin{vmatrix} 1 & \alpha & \alpha^2 & -\beta\gamma \\ 1 & \beta & \beta^2 & -\gamma\alpha \\ 1 & \gamma & \gamma^2 & -\alpha\beta \end{vmatrix} = 0$

(c) If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$. 5

10.(a) Show that $\nabla r^n = nr^{n-2}\vec{r}$, where $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ 5

(b) Evaluate $\iint\sqrt{4x^2 - y^2}\,dxdy$ over the triangle formed by the straight lines $y = 0$, $x = 1$ and $y = x$. 5

(c) Verify Stokes theorem for $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. 5

11.(a) Show that $\vec{f} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Hence find a scalar function ϕ such that $\vec{f} = \nabla\phi$.

(b) Given that

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{for } (x, y) \ne (0,0) \\ 0, & \text{for } (x, y) = (0,0) \end{cases}$$

Show that $f_{xy}(0,0) \ne f_{yx}(0,0)$.

(c) Evaluate

$$\int_C (3xydx - y^2dy)$$

Where C is the arc of the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.