Name : ....

Roll No. : ....

Invigilator's Signature : ....

CS / INTPBIR / SEM-2 / PHY-202 / 2011

2011

MATHEMATICAL METHODS – II

Time Allotted : 3 Hours *Full Marks* : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer *Question No. 1* and any *two* from the rest.

1. a) Show that the system of matrices of the special form

$$\begin{pmatrix} x + y\tan\theta & -\frac{y}{\lambda}\sec\theta \\ \lambda y\sec\theta & x - y\tan\theta \end{pmatrix}$$

where $\lambda \in \mathcal{R}$ and $\neq 0$, $\theta \neq \frac{\pi}{2}$ are given and x, y are any two real numbers, $x^2 + y^2 \neq 0$, form an Abelian group with respect to matrix multiplication. Hence demonstrate that the above system of matrices, combined by matrix addition and multiplication, is isomorphic to the field of complex numbers $z = x + iy$.

9

b) For $u \in SU(2)$ associate a transformation $X \to u h_X u^{\dagger}$ where $h_X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ and $X = (x_1, x_2, x_3)$ in $\mathcal{R}^3$. Show with the help of this transformation that the $SU(2)$ matrix $\cos\frac{t}{2} I + i \sin\frac{t}{2}\sigma_1$ corresponds to rotation about the x_1-axis through an angle t in $\mathcal{R}^3$. 9

2. a) Show that the Fourier transform of the function $\frac{1}{x(x^2 + a^2)}$, $a > 0$, is given by :

$$\mathcal{F}\left[\frac{1}{x(x^2 + a^2)}, \xi\right] = i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{a^2}\left(1 - e^{-|\xi|a}\right) sgn\ \xi . \qquad 9$$

Hence show that the Fourier Sine transform for the above function is given by :

$$\mathcal{F}_s\left[\frac{1}{x(x^2 + a^2)}, \xi\right] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{a^2}\left(1 - e^{-|\xi|a}\right) sgn\ \xi . \qquad 3$$

b) Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$

where $ad - bc \neq 0$ is a rational number. Prove that G forms a group under matrix multiplication. 4

3. a) Find a general expression for the variation of the action functional : 8

$S[\phi_\alpha, \partial_\mu \phi_\alpha] = \int_\Omega dx_\mu \mathcal{L} \ (\phi_\alpha, \partial_\mu \phi_\alpha, x_\mu)$ where Ω is a region in the x_μ space, $\alpha = 1, 2, \ldots. N$.

b) Obtain Noether's identity. 4

c) Deduce the equation for conservation of the energy-momentum tensor when the Lagrangian density $\mathcal{L}(\phi_\alpha, \partial_\mu \phi_\alpha)$ is explicitly independent of x_μ and the total variation of ϕ_α *viz.* $\delta\phi_\alpha = 0$. Deduce the expression for the 4-momentum density and hence write down the expression for the Hamiltonian density. 4

4. a) Find the solution to the two-dimensional Laplace's equation :

$\partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0$ in the half plane $y \geq 0$, with the boundary condition :

$u(x, 0) = f(x), \ -\infty < x < \infty$ and the limiting condition

$u(x, y) \to 0$ as $\rho = \left(x^2 + y^2\right)^{\frac{1}{2}} \to \infty$. 10

b) Define Lorentz transformations in four dimensions and show that they form a group. 6

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